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STRONG COMPACTNESS AND OTHER CARDINAL SINS

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0. Introduction

Our notation and terminology will conform to that used in the most recent set-theoretic literature. For example, cardinals are initial ordinals. If A is a set, $|A|$ denotes the cardinality of A. For unfamiliar items, we urge the reader to consult Mathias [16]. The results of this paper are intended to be developed in **ZFC**, that is, the Zermelo–Fraenkel set theory with the axiom of choice. Parts of this paper are contained in the author's doctoral dissertation (Ketonen [7]).

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At the referees request we include a brief summary of our results.

In Section 0, we introduce our basic notions; those of countable completeness, strong compactness, supercompactness and orderings involving ultrafilters.

Section 1 includes our basic results, Theorem 1.4, which gives a complete description of the degree of regularity of a given ultrafilter satisfying the requirement of countable completeness in terms of ordinals of the ultrapower. As a consequence of this result we show how to get non-regular ultrafilters over ω_2 from countably complete ones.

Section 2 concerns itself with the basic properties of the Keisler-order on countably complete ultrafilters and the use of ordinals of the ultrapowers in question to describe this. We wish to remark in this connection that Robert Solovay has proved the above-mentioned order to be well-founded.

Section 3 introduces the notion of unique regularity and its connection with supercompactness. Section 4 gets more to the point; we study the connection between regularity and certain simple sets belonging to normal ultrafilters over power-sets of ordinals (Theorem 4.2 etc.). We describe another embedding axiom which yields the existence of non-trivial non-regular ultrafilters.

Section 5 gives some results on products and sums of ultrafilters and the effect of these operations on regularity. These facts and Theorem 1.4 yield a nice and simple characterization of strong compactness (Theorem 5.9). Since we consider this result the most important of this paper, we include another proof of this fact which uses an interesting partial ordering on ultrafilters.

The following notions will be of fundamental importance in this paper

Definition 0.1. Let κ, λ, μ be cardinals so that $\kappa \leq \mu$. An ultrafilter D over λ is (κ, μ)-*regular* if there is a family $\{X_\alpha \mid \alpha \in \mu\}$ of elements of D so that for every subset S of μ of cardinality κ, $\bigcap_{\alpha\in S} X_\alpha = 0$.

For more on this notion, see Benda [1, 2] and Keisler [6].

The following notion is due to H.J. Keisler:

Definition 0.2. Let U, D be ultrafilters over the cardinals μ, λ, respectively. Then D is *projectible* onto U, $U \leq D$, if there is a function $f\colon \lambda \to \mu$ so that for every $X \subseteq \mu$,

$$X \in U \leftrightarrow f^{-1}(X) \in D\,.$$

We denote $U = f^*(D)$. We say that U is *isomorphic* to D, $U \cong D$, if $U \leq D$ and $D \leq U$.

For more on this order, see Kunen [11] and Ketonen [9]. The above situation can be described in terms of elementary embeddings. Let U, D be countably complete ultrafilters over a cardinal κ. Let i_U, i_D be the induced embeddings of V into transitive submodels M_U, M_D of the universe.

Proposition 0.3. (1) *If $U \leq D$, then there is an elementary embedding $\kappa\colon M_U \to M_D$ that $\kappa \circ i_U = i_D$.*

(2) *$U \leq D$ if and only if there is a set $X \in P(i_U(\kappa)) \cap M$ which is represented in the ultrapower by a function $A : \kappa \to P(\kappa)$ so that $\{A(\alpha) \mid \alpha < \kappa\}$ is a disjointed family and there is an ultrafilter V over $P(i_U(\kappa)) \cap M$, not necessarily belonging to M, so that M_D is the transitive model isomorphic to the ultrapower of M_U by V and $V \ni X$, $i_D = i_V \circ i_U$.*

Proof. For if $U = f^*(D)$, define $A : \kappa \to P(\kappa)$ by $A(\alpha) = f^{-1}(\{\alpha\})$. Define an ultrafilter $\mathcal{U}$ over $[A]$ in the ultrapower V^κ/U by

$$[X] \in \mathcal{U} \leftrightarrow \bigcup_{\alpha<\kappa} X(\alpha) \in D .$$

This proves the proposition.

It is actually easy to check that this $\mathcal{U} \in M$ if and only if D is isomorphic to a U-sum of ultrafilters.

Definition 0.4. Let U, D_α $(\alpha < \kappa)$ be ultrafilters over κ, λ_α $(\alpha < \kappa)$, respectively. Then the *U-sum* of D_α's, $\Sigma_U D_\alpha$, is the set of all subsets X of $\Gamma = \{(\alpha, \beta) \mid \alpha < \kappa, \beta < \lambda_\alpha\}$ so that

$$\{\alpha \mid \; X \mid \alpha \in D_\alpha\} \in U ,$$

where $X \mid \alpha = \{\beta \mid (\alpha, \beta) \in X\}$.

We could, of course, consider a more general operation, "fusion". Define an ultrafilter V by

$$X \in V \leftrightarrow \{\alpha \mid X \cap \lambda_\alpha \in D_\alpha\} \in U .$$

Obviously, this always defines an ultrafilter. Then any U-sum of ultrafilters is isomorphic to a fusion of ultrafilters, but the converse does not hold.

In the special case of all D_α's being the same we get the definition of a product of two ultrafilters.

Definition 0.5. Let U, V be ultrafilters over κ, λ, respectively. Then

$$U \times V = \{X \subseteq \kappa \times \lambda \mid \{\alpha \mid \; X \mid \alpha \in V\} \in U\} .$$

For more on products, see Ketonen [9]. It is easy to prove the following.

Proposition 0.6. *Let U, V be ultrafilters. Then:*

(1) $U \le U \times V$, $V \le U \times V$. *In fact, if both U, V are non-principal, then* $U < U \times V$, $V < U \times V$.

(2) *If both U, V are κ-complete, so is $U \times V$.*

(3) *If $\kappa \le \lambda$ are cardinals, and U or V is (κ, λ)-regular, so is $U \times V$.*

There is a natural generalization of the above operation: Suppose that U, V, D are ultrafilters over λ, μ, ρ, respectively, such that $U, V \le D$. We have functions f, g so that $U = f^*(D)$ and $V = g^*(D)$. Define a function $h : \rho \to \lambda \times \mu$ by setting $h(\alpha) = (f(\alpha), g(\alpha))$ for $\alpha < \rho$. Define the ultrafilter $U \oplus V$ over $\lambda \times \mu$ by setting

$$X \in U \oplus V \leftrightarrow h^{-1}(X) \in D .$$

Another way of thinking about this operation: U corresponds to the partitioning $\{f^{-1}(\delta) \mid \delta < \lambda\}$, V to $\{g^{-1}(\{\delta\}) \mid \delta < \mu\}$ of ρ, and $U \oplus V$ to their common refinement. The result of '$\oplus$' depends heavily on the choice of the projecting functions: We can get both $U \times V$ and $V \times U$ in some situations.

Definition 0.7. A function $f : \lambda \to \lambda$ is *pressing down* if

$$\alpha \ge \omega \to f(\alpha) < \alpha .$$

Let P_λ = set of all pressing down functions on λ.

Definition 0.8. An ultrafilter D over a regular cardinal λ is *weakly normal* if every pressing down function f on λ is bounded by a constant $< \lambda \pmod{D}$.

For any $F : P_\lambda \to \lambda$ and cardinal $\kappa \le \lambda$, let T_F^κ be the filter κ-generated by the sets

$$\{\{\alpha \mid f(\alpha) \le F(f)\} \mid f \in P_\lambda\} .$$

Proposition 0.9. (1) *An ultrafilter D over λ is weakly normal if and only if for some $F: P_\lambda \to \lambda$, $T_F^\omega \subseteq D$.*

(2) *If $F, G: P_\lambda \to \lambda$ so that $F \leq G$ (i.e. for any $f \in P_\lambda$, $F(f) \leq G(f)$) then for any $\kappa \leq \lambda$, $T_F^\kappa \supseteq T_G^\kappa$.*

(3) *If U, D ultrafilters and $U \leq D$, then*

(a) *if U is (κ, λ)-regular, so is D;*

(b) *if D is κ-complete, so is U.*

(4) *If D is uniform κ-complete over λ, $\kappa \geq \omega_1$, then there is a uniform κ-complete weakly normal ultrafilter U over λ so that $U \leq D$.*

(5) *If D is uniform weakly normal over λ, then D is minimal among the ultrafilters over λ in the $\leq$-order; i.e., if $f: \lambda \to \lambda$, then either f is 1–1 or bounded on a set belonging to D.*

(6) *If λ is measurable so that any λ-complete filter over λ can be extended to a λ-complete ultrafilter, then for any $F: P_\lambda \to \lambda$, if $T_F^\lambda \not\ni 0$, then T_F^λ is the intersection of $< \lambda$ normal ultrafilters over λ.*

(7) *Assume that the hypotheses of (6) hold and that there is only one normal ultrafilter D over λ. Then, if M is a transitive model of* **ZFC** *so that $M \cap P(\lambda) = P(\lambda)$ and every function $F: 2^\lambda \to \lambda$ is bounded by a function $G: 2^\lambda \to \lambda$ so that $G \in M$, then $D \in M$.*

Proof. (4). Let F be the first function $(\operatorname{mod} D)$ so that $f > \xi \,(\operatorname{mod} D)$ for every $\xi < \lambda$. Then let $U = f^*(D)$.

(5). Given $f: \lambda \to \lambda$, define

$$g(\alpha) = \mu\beta[f(\beta) \geq f(\alpha)] \leq \alpha .$$

If G is pressing down $(\operatorname{mod} D)$, select a $\xi < \lambda$ such that for every $\alpha < \lambda$, $g(\alpha) \leq \xi$ on a set $A \in D$, and let

$$\eta = \sup\{f(\alpha) \mid \alpha \leq \xi\} < \lambda .$$

Then $f \leq \eta$ on A. If g is not pressing down, there is a $B \in D$ so that $g(\alpha) = \alpha$ for $\alpha \in \beta$. Thus for every $\alpha, \beta \in B$, if $\alpha < \beta$, then $f(\alpha) < f(\beta)$, i.e., f is strictly increasing.

(6). If $T_F^\lambda \not\ni 0$, any λ-complete extension of T_F^λ is a normal ultrafilter. If T_F^λ would have λ distinct extensions to normal ultrafilters, $\{D_\alpha \mid \alpha < \lambda\}$, then we can find a disjointed family $\{X_\alpha \mid \alpha < \lambda\}$ of subsets of λ such that for every $\alpha < \lambda$, $T_F^\lambda \cup \{X_\alpha\}$ generates a non-trivial λ-complete ultrafilter: Take $X_{\alpha\beta} \in D_\alpha \setminus D_\beta$, for any $\alpha, \beta < \lambda$, $\alpha \neq \beta$, let $y_\alpha \in D_\alpha$

such that for any $\beta \neq \alpha$, $|y_\alpha \setminus X_{\alpha\beta}| < \lambda$, and let

$$X_\alpha = y_\alpha \setminus \bigcup_{\beta<\alpha} y_\beta \qquad (\alpha < \lambda).$$

Thus, if D is normal, the filter generated by

$$T_F^\lambda \cup \left\{ \bigcup_{\alpha \in C} X_\alpha \mid C \in E \right\},$$

where E is an ultrafilter over κ isomorphic to $D \times D$, can be extended to a normal ultrafilter V. But $D \times D \leq V$. This contradicts (5) and Proposition 0.6.

The regularity of λ is quite essential in the proof of (5): The U we construct is in general isomorphic to a weakly normal ultrafilter over $\operatorname{cof}(\lambda)$. In general, any weakly normal ultrafilter over λ is concentrated on a set of cardinality $\operatorname{cof}(\lambda)$. This difficulty can be circumvented in the following fashion. Instead of the function f, at first take f_0 such that for any $X \in D$, $|f_0''(X)| = \lambda$. Then the resulting uniform "weakly normal" ultrafilter U has the following properties: If $f < \text{id} \pmod U$, then there is a $X \in U$ so that $|f''(X)| < \lambda$. If $f \geq \text{id}$, then f is $1-1$ on a set $X \in U$. Consequently, U is $\leq$-minimal among the uniform ultrafilters over λ. Also, any set $X \in U$ is "Mahlo" in the following sense: If f is pressing down on X, then there is a set $Y \subseteq X$ of cardinality λ so that $|f''(Y)| < \lambda$.

Definition 0.10. A regular cardinal $\kappa > \omega$ is *strongly compact* if every κ-complete filter over any set can be extended to a κ-complete ultrafilter.

Definition 0.11. A regular cardinal $\kappa > \omega$ is *supercompact* if for every cardinal $\alpha > \kappa$ there is an elementary embedding j from V into a transitive submodel M such that $j(\kappa) > \kappa$ and $j \mid R(\kappa) = \text{id}$ and $M^\alpha \subseteq M$; i.e., M is closed under α-sequences.

For more on strongly compact and supercompact cardinals, see Kunen [11, 12], Solovay–Reinhardt [21] and Vopěnka–Hrbáček [22]. The following two results will be used.

Theorem 0.12. *If $\kappa > \omega$ is a regular cardinal, then κ is strongly compact if and only if for every $\lambda > \kappa$ there is a κ-complete, (κ, λ)-regular ultrafilter.*

This is a fairly well-known fact and is proved using standard model-theoretic methods of, for example, Chang–Keisler [3].

Definition 0.13. An ultrafilter D over $P(\lambda)$ is *normal* if every function F which is pressing down on $P(\lambda)$ (i.e., for a.e. A, $F(A) \in A$) is constant $(\text{mod}\, D)$.

Theorem 0.14. (Solovay–Reinhardt). *If κ is a regular cardinal $> \omega$, then κ is supercompact if and only if for every regular $\lambda > \kappa$ there is a κ-complete normal ultrafilter over $S_\kappa(\lambda) = \{X \subseteq \lambda \mid |X| < \kappa\}$ which does not contain any $S_\kappa(\xi)$ for $\xi < \lambda$.*

1. Criteria for regularity

We shall assume throughout this section that κ, λ are regular cardinals such that $\kappa \leq \lambda$.

Proposition 1.1. *Suppose D is a (κ, λ)-regular ultrafilter over λ and F a function from λ to λ so that for every $\alpha < \lambda$, $\operatorname{cof}(F(\alpha)) \geq \kappa$. Then*

$$\operatorname{cof}(\Pi_D \langle F(\alpha), < \rangle) \geq \lambda^+ .$$

Proof. For if we are given functions $\{G_\alpha \mid \alpha < \lambda\}$ from λ to λ so that $G_\alpha < F \pmod D$, if $\{X_\gamma \mid \gamma < \lambda\}$ is a (κ, λ)-regularizing family for D, and if we define

$$G(\delta) = \sup \{G_\gamma(\delta) \mid \delta \in X_\gamma\} \qquad (\delta < \lambda) ,$$

then $G < F \pmod D$, and for every $\gamma < \lambda$, $G_\gamma \leq G$ on $X_\gamma \in D$.

Theorem 1.2. *Suppose D is a countably complete uniform ultrafilter over λ, and f is the first function greater than all constants $< \lambda \pmod D$. Then, if the set $X = \{\alpha \mid \operatorname{cof}(f(\alpha)) \geq \kappa\} \in D$, we have: There is a function F from λ to λ so that for every α, $F(\alpha)$ is a regular cardinal $\geq \kappa$ and*

$$\operatorname{cof}(\Pi_D \langle F(\alpha), < \rangle) = \lambda.$$

Proof. Define

$$F(\alpha) = \begin{cases} \operatorname{cof}(f(\alpha)) & \text{if } \operatorname{cof}(f(\alpha)) \geq \kappa , \\ \kappa & \text{otherwise} . \end{cases}$$

Then

$$V^\lambda/D \models [F]_D = \operatorname{cof}([f]_D) ,$$

so that

$$V^\lambda/D \models \operatorname{cof}([F]_D) = \operatorname{cof}([f]_D) = \lambda .$$

For brevity, we shall call f the first function of D in the situation described in the statement of Theorem 1.2.

Theorem 1.3. *Suppose D is a uniform countably complete ultrafilter over λ, and f its first function. Then, if $Y = \{\alpha \mid \operatorname{cof}(F(\alpha)) < \kappa\} \in D$, D is (κ, λ)-regular.*

Proof. Let $A_\alpha \subseteq f(\alpha)$ be a cofinal set in $f(\alpha)$ so that $|A_\alpha| < \kappa$ for every α in Y. Let j be the elementary embedding

$$V \to V^\lambda/D \overset{\text{def}}{=} M$$

associated with D. Then $[f]_D$ (the D-equivalence class of f) represents the supremum of ordinals $j(\xi)$, $\xi < \lambda$, in M. Define a function $F\colon \lambda \to V$ by setting $F(\alpha) = A_\alpha$ for $\alpha \in Y$. Then we have

$$M \models [F] \text{ is a cofinal set in } [f] .$$

Define an 'increasing' sequence of disjointed intervals $I_\eta = (\xi_\eta, \mu_\eta)$, $\xi_\eta < \mu_\eta < \lambda$, by induction as follows:

Given $\{I_\eta \mid \eta < \eta'\}$, let

$$\xi_{\eta'} = (\sup_{\eta < \eta'} \mu_\eta) + 1 ,$$

and let $\mu_{\eta'}$ be the least ordinal $\mu < \lambda$ so that

$$M \models [F] \cap (j(\xi_{\eta'}), j(\mu)) \neq 0 .$$

For $\eta < \lambda$, define

$$X_\eta = \{\alpha \in Y \mid A_\alpha \cap I_\eta \neq 0\} .$$

Obviously, every $X_\eta \in D$, and the family $\{X_\eta \mid \eta < \lambda\}$ regularizes D: If S is a subset of λ of cardinality κ, then

$$\bigcap_{\eta \in S} X_\eta = \{\alpha \in Y \mid \eta \in S, A_\alpha \cap I_\eta \neq 0\} = 0 ,$$

since for $\alpha \in Y$ we have $|A_\alpha| < \kappa$, and the I_η are disjointed.

As K. Prikry pointed out, we need only assume the existence of the first function of D; the same proof carries through. Theorems 1.1–1.3 yield our basic result, Theorem 1.4.

We denote the ultrapower V^λ/D sometimes by V^D. We trust that this will not lead to confusion.

Theorem 1.4. *Let D be a uniform, countably complete ultrafilter over λ, and f its first function. Then the following are equivalent:*

(1) *D is (κ, λ)-regular.*

(2) *In V^D*, $\mathrm{cof}([f]) < j(\kappa)$; *i.e.*, $\{\alpha \mid \mathrm{cof}(f(\alpha)) < \kappa\} \in D$.

(3) *There is no function $F: \lambda \to \lambda$ so that for every $\alpha < \lambda$, $F(\alpha)$ is a regular cardinal $\geq \kappa$ and* $\mathrm{cof}(\Pi_D \langle F(\alpha), < \rangle) = \lambda$.

In the special case $\lambda = \kappa^+$ this theorem reads:

Theorem 1.5. *Let D be a uniform, countably complete ultrafilter over κ^+, and f its first function. Then the following are equivalent:*

(1) *D is not (κ, κ^+)-regular.*

(2) *In V^D*, $\mathrm{cof}([f]) = j(\kappa)$.

(3) $\mathrm{cof}(\Pi_D \langle \kappa, < \rangle) = \kappa^+$.

An easy extension of the method of proof of Theorem 1.4 gives:

Theorem 1.6. *Let D be a countably complete uniform ultrafilter over λ, and for $\mu \leq \lambda$, let f_μ be the first function greater than all the constants $< \mu$. If μ is a regular cardinal, then D is (κ,μ)-regular if and only if* $\mathrm{cof}([f_\mu]) < j(\kappa)$ *in V^D. Thus, D is (κ, λ)-regular if and only if for every regular $\mu \leq \lambda$ we have* $\mathrm{cof}([f_\mu]) < j(\kappa)$ *in V^D if and only if* $\mathrm{cof}([f_\lambda]) < j(\kappa)$ *in V^D.*

As a corollary to Theorem 1.6 we get:

Theorem 1.7. *Let D be a κ-complete uniform ultrafilter over λ. Let σ be a singular cardinal of cofinality $< \kappa$ so that $\kappa \leq \sigma < \lambda$. Then for some cardinal $\mu < \sigma$, D is (μ, σ^+)-regular.*

Proof. There is a $\mu < \sigma$ so that in V^D, $\mathrm{cof}([f_{\sigma^+}]) < \mu$; this proves the theorem.

K. Kunen pointed out that using the results of Kunen and Prikry [15] we get from Theorem 1.7:

Theorem 1.8. *Let D be a κ-complete uniform ultrafilter over λ. Then, if for some cardinal $\kappa \leq \mu \leq \lambda$, D is μ-descendingly complete, there exists a cardinal σ of cofinality $\geq \kappa$ so that $\kappa \leq \sigma \leq \mu$ and D is σ-descendingly complete. In particular, if $\lambda < \omega_{\kappa+\kappa}$, D is μ-descendingly incomplete for every μ between κ, λ.*

From regular ultrafilters we can get weakly normal, hence minimal, ones.

Theorem 1.9. *Let D be a countably complete uniform ultrafilter over λ and let U be defined by*

$$X \in U \leftrightarrow f^{-1}(X) \in D .$$

Then U is (κ, λ)-regular if and only if D is if and only if

$$\{\alpha \mid \operatorname{cof}(\alpha) < \kappa\} \in U .$$

Proof. If U is not (κ, λ)-regular, then

$$\{\alpha \mid \operatorname{cof}(f(\alpha)) \geq \kappa\} = f^{-1}(\{\alpha \mid \operatorname{cof}(\alpha) \geq \kappa\}) \in D .$$

It is worth noting that the existence of a non-(κ,κ^+)-regular countably complete ultrafilter implies the existence of a non-regular ultrafilter in a boolean extension of the universe. For let B be the standard Levy–Solovay algebra for collapsing κ onto ω_1. (For details, see Solovay [20].) Then B has κ-C.C.C., and we have:

Theorem 1.10. *Suppose κ is strongly inaccesssible and D countably complete uniform over κ^+ so that D is not (κ, κ^+)-regular. Then in V^B, D is a filter so that any extension of D to an ultrafilter U is non-regular and* $\operatorname{cof}(\Pi_U \langle \omega_1, < \rangle) = \omega_2$.

Note that in V^B, κ^+ becomes ω_2, so that in fact we can have such a U uniform over ω_2.

Proof. Let $\{F_\gamma \mid \gamma < \kappa^+\}$ be a cofinal sequence in $\Pi_D \langle \kappa, < \rangle$ and let

$U \supseteq D$ be an ultrafilter in V^B with value 1. If F is a function $\kappa^+ \to \kappa$ with value 1 in V^B, by κ-C.C.C. there is a standard function $G: \kappa^+ \to \kappa$ so that $F \leq G$ with value 1. Hence, there is a $\gamma < \kappa^+$ so that $F \leq F_\xi \pmod D$, hence $\pmod U$, with value 1.

We shall now give some further criteria for regularity.

Definition 1.11. An ultrafilter D over $P(\lambda)$ is *weakly normal* if every pressing down function F is bounded $\pmod D$, i.e., if for almost every A in $P(\lambda)$, $F(A) \in A$, then there is a $\xi < \lambda$ so that $F \leq \xi$ almost everywhere.

Theorem 1.12. *Suppose D is a countably complete ultrafilter over $P(\lambda)$ so that for every $\xi < \lambda$, $P(\xi) \notin D$, and suppose that D is weakly normal. Then D is (κ, λ)-regular if and only if*

$$\bar{S}_\kappa(\lambda) \stackrel{\text{def}}{=} \{A \subseteq \lambda \mid \operatorname{cof}(A) < \kappa\} \in D\,.$$

Proof. Obvious from Theorem 1.6, since the function $g(A) \stackrel{\text{def}}{=} \sup(A)$ represents the first function greater than every constant $< \lambda$.

Theorem 1.13. *Let D be a countably complete uniform ultrafilter over λ. Then the following statements are equivalent:*

(1) *D is (κ, λ)-regular.*

(2) *There is a function $m: \lambda \to S_\kappa(\lambda)$ so that for every $\xi < \lambda$,*

$$\{A \mid \xi \in m(\eta)\} \in D\,.$$

(3) *There is a function $m: \lambda \to S_\kappa(\lambda)$ so that for every $\delta > \lambda$, $m(\delta) \subseteq \delta$, and for every $X \in D$,*

$$\bigcup \{m(\delta) \mid \delta \in X\} = \lambda\,.$$

(4) *There is a weakly normal ultrafilter U over $S_\kappa(\lambda)$ projectible to D so that for every $\xi < \lambda$ we have: $\{A \mid \xi \in A\} \in U$.*

(5) *There is an element $X \in V^D$ so that in V^D*, card $(X) < j(\kappa)$, *and for every $\xi < \lambda$, $j(\xi) \in X$ in V^D, where j is the elementary embedding into V^D associated with D.*

(6) *Given any* λ *elements* X_ξ $(\xi < \lambda)$ *in* V^D, *there is an* $X \in V^D$ *so that* $\operatorname{card}(X) < j(\kappa)$ *and each* X_ξ *belongs to* X *in* V^D.

Proof. If D is regularized by $\{X_\gamma \mid \gamma < \lambda\}$, let $m(\delta) = \{\gamma \mid \delta \in X_\gamma\}$ for $\delta < \lambda$. In (4), define U by

$$X \in U \leftrightarrow m^{-1}(X) \in D.$$

In (3), note that we can, of course, assume that $X_\gamma \subseteq (\gamma, \lambda) = \{\alpha \mid \gamma < \alpha < \lambda\}$. In (7), if f_ξ is the function $\lambda \to V$ representing X_ξ, define for $\delta < \lambda$,

$$X(\delta) = \{f_\xi(\delta) \mid \xi \in m(\delta)\} .$$

2. The Keisler-ordering

We wish to make some brief remarks on the structure of the ordering $\leq$ on countably complete ultrafilters. As before, κ, λ will denote regular cardinals with $\kappa \leq \lambda$.

Theorem 2.1. *Let D be a uniform, countably complete, (κ,λ)-regular ultrafilter over λ. Then D is μ-universal for every cardinal μ so that $2^{\mu} \leq \lambda$ in the following sense: If U is any κ-complete ultrafilter over μ, then $U \leq D$.*

Proof. Let μ, U be as in the statement of the theorem. Let i be the induced elementary embedding $V \to M$ into the transitive submodel isomorphic to V^D. Then $M \models i(U)$ is $i(\kappa)$-complete over $i(\mu)$, and for every $X \in U$, $i(X) \in i(U)$. Thus by Theorem 1.14.(6) there is a filter $\mathcal{F}$ in M so that for every $X \in U$,

$$M \models i(X) \in \mathcal{F} \subseteq i(U) ,$$

and so that

$$M \models \operatorname{card}(\mathcal{F}) < i(\kappa) .$$

Hence

$$M \models \bigcap \mathcal{F} \neq 0 .$$

Let δ be any ordinal so that $\delta \in \bigcap \mathcal{F}$. Then δ corresponds to a function $F: \lambda \to \mu$. Obviously, for every $X \in U$, $\delta \in j(X)$. Hence, $F^{-1}(X) \in U$. Hence, for any $X \in U$,

$$X \in U \leftrightarrow F^{-1}(X) \in D.$$

The above proof yields a natural correspondence between ordinals δ so that $\mu_0 = \sup\{i(\xi) | \xi < \mu\} \leq \delta < i(\mu)$ and ultrafilters $D_\delta = \{X \subseteq \mu | \delta \in i(X)\}$. Moreover, any κ-complete uniform ultrafilter over μ is a D_δ. If h_δ is the first function of D_δ, then

$$(*) \qquad (ih_\delta)(\delta) = m(\delta) ,$$

where

$$m(\delta) = \min\{(ih)(\delta) | h \in {}^{\mu}\mu, (ih)(\delta) \geq \mu_0\} .$$

Conversely, if $h \in {}^{\mu}\mu$ satisfies (*), h is the first function of D_δ. This gives a characterization of all non-(κ,μ)-regular ultrafilters in terms of ordinals in M:

Proposition 2.2. (1) *Let Γ be the set of all $h \in {}^{\mu}\mu$ so that h appears as a first function of some κ-complete ultrafilter over μ. Then*

$$\Gamma = \{h \in {}^{\mu}\mu \mid \exists \delta\colon (ih)(\delta) = m(\delta)\}\ .$$

(2) *There is a non-(κ, μ)-regular ultrafilter over μ if and only if $i(\kappa) = \mathrm{cof}(m(\delta))$ for some $\delta \geq \mu_0$ if and only if there is a $\delta \geq \mu_0$ so that $\mathrm{cof}(\delta) = i(\kappa)$ and for every $h \in {}^{\mu}\mu$ either $(ih)(\delta) < \mu_0$ or $(ih)(\delta) \geq \delta$.*

(3) *D_δ is weakly normal if and only if $m(\delta) = \delta$.*

The following theorem is due to K. Kunen.

Theorem 2.3. *Assume that κ is strongly compact and $\lambda^{\kappa} = \lambda$. Then:*

(1) *For any $\alpha < (2^\lambda)^+$ there is a κ-complete uniform ultrafilter D_α over λ so that $i_{D_\alpha}(\kappa) > \alpha$, where i_{D_α} is the induced elementary embedding of V into the transitive submodel isomorphic to V^{D_α}.*

(2) *Given any family of 2^λ κ-complete ultrafilters over λ, $\{U_\alpha \mid \alpha < 2^\lambda\}$ there is a κ-complete uniform ultrafilter U so that for every $\alpha < 2^\lambda$, $U_\alpha \leq U$.*

Proof. For (1), see Kunen [13]. Part (2) is proved similarly. Let $\{\theta_\alpha \mid \alpha < 2^\lambda\}$ be a κ-independent family of functions $\lambda \to \lambda$, and let U be any κ-complete uniform ultrafilter extending the κ-complete filter

$$\{\theta_\alpha^{-1}(X) \mid X \in U_\alpha, \alpha < 2^\lambda\}\ .$$

Now it is well-known that for any κ-complete U over λ, $i_U(\kappa) < (2^\lambda)^+$ (see Kunen [10]). Also, if D, U are κ-complete over λ with $D \leq U$, then $i_D(\kappa) \leq i_U(\kappa)$. For if $f\colon \lambda \to \lambda$ is the projecting function giving D from U, then the map $g \to g \circ f$ induces an elementary embedding $V^D \to V^U$ mapping ordinals less than the constant function κ to ordinals with the same property. Hence, if $\{D_\alpha \mid \alpha < (2^\lambda)^+\}$ is the family of ultrafilters given by Theorem 2.3 (1), there is no D over λ so that for every $\alpha < (2^\lambda)^+$, $D_\alpha \leq D$. Thus, in a sense, Theorems 2.1 and 2.3.(2) are the best possible.

3. Uniquely regular ultrafilters

Let κ, λ be regular cardinals with $\kappa \leq \lambda$. We shall investigate the following, rather pathological, concept:

Definition 3.1. An ultrafilter D over λ is *uniquely* (κ, λ)*-regular* if it has a (κ,λ)-regularizing family $\{X_\alpha | \alpha < \lambda\}$ so that for any other (κ, λ)-regularizing family $\{Y_\alpha | \alpha < \lambda\}$ for D there is a $Y \in D$ so that for every $\alpha < \lambda$, $Y \cap X_\alpha \subseteq Y_\alpha$.

Proposition 3.2. (M. Benda). *If D is a κ-complete non-principal normal ultrafilter over κ, then D is uniquely* (κ,κ)*-regular.*

Proof. For then $X_\xi = (\xi, \kappa)$,$\xi < \kappa$, is the uniquely regularizing family. For if $\{Y_\xi | \xi < \kappa\}$ is a decreasing sequence of elements of D so that their intersection is empty, define a function $g: \kappa \to \kappa$ by setting $g = \xi$ on

$$\bigcap_{\eta<\xi} Y_\eta \setminus Y_\xi$$

for $\xi < \kappa$. Then $Y = \{\xi | g(\xi) \geq \xi\} \in D$. Hence, if $Y \in X_\xi \cap Y$, $\gamma > \xi$, so $g(\gamma) \geq \gamma > \xi$; i.e.,

$$\gamma \in Y_{g(\gamma)} \subseteq Y_\xi \ .$$

In practice it is often very difficult to verify the unique regularity of a given ultrafilter.

Example. Assume κ is supercompact, let $j: V \to M$ be an elementary embedding of V into a transitive submodel M so that $j(\kappa) > U$ and $j | R(\kappa) = \text{id}$. If $A \subseteq \lambda$ is of cardinality λ, λ regular, then the ultrafilter

$$D_A = \{X \subseteq S_\kappa(\lambda) | G^A \in j(X)\} ,$$

where

$$G^A = \{j(\xi) | \xi \in A\} ,$$

is normal κ-complete over $S_\kappa(\lambda)$. Define a new ultrafilter U over $S_\kappa(\lambda)$ by

$$X \in U \leftrightarrow \{A \mid X \in D_{\lambda \setminus A}\} \in D.$$

Then U is κ-complete uniform over $S_\kappa(\lambda)$ so that no pressing down function is constant. Also, for $S \subseteq \lambda$,

$$\bigcup_{\xi \in S} P_\xi \in U \leftrightarrow |S| \geq \kappa,$$

where we denote here and henceforth

$$P_\xi = \{A \mid \xi \in A\}.$$

Hence, the family

$$Q_\alpha = \bigcup \{P_\alpha \mid \lambda \cdot \kappa < \alpha < \lambda \cdot \kappa + \kappa\}$$

regularizes U. Is U uniquely regular?

Proposition 3.3. *Let D be an ultrafilter over $S_\kappa(\lambda)$ so that there exists a map m: $S_\kappa(\lambda) \to S_\kappa(\lambda)$ so that for every function $G : S_\kappa(\lambda) \to \lambda$: If for a.e. A, $G(A) \in m(A)$, then G is constant* (mod D).

Let $X_\xi = \{A \mid \xi \in m(A)\}$. Then, for every family $\{T_\xi \mid \xi < \lambda\} \subseteq D$ there is a $X \in D$ so that for $\xi < \lambda$,

$$X \cap X_\xi \subseteq T_\xi.$$

Proof. For $A \in S_\kappa(\lambda)$, define

$$p(A) = \{\xi \mid A \in T_\xi\}.$$

Case 1. $\{A \mid m(A) \nsubseteq p(A)\} \in D$. In this case, define

$$G(A) = \mu\xi\ [\xi \in m(A) \text{ and } \xi \notin p(A)].$$

Then for a.e. A, $G(A) \in m(A)$. Hence G is a constant, say ξ_0, on a set $Y \in D$. Thus for $A \in Y$, $\xi_0 \notin p(A)$; i.e., $A \notin T_{\xi_0}$. Hence $Y \cap T_{\xi_0} = 0$, a contradiction. Hence we must have:

Case 2. $Q \stackrel{\text{def}}{=} \{A \mid m(A) \subseteq p(A)\} \in D$. Then for $A \in Q$,

$$\xi \in m(A) \to \xi \in p(A);$$

i.e., if $A \in X_\xi$, then $A \in T_\xi$. Hence for every $\xi < \lambda$, $Q \cap X_\xi \subseteq T_\xi$.

Proposition 3.4. *Suppose D is an ultrafilter over $S_\kappa(\lambda)$ with $\{X_\xi | \xi < \lambda\}$ as a (κ, λ)-uniquely regularizing set for it. Define a function $m: S_\kappa(\lambda) \to S_\kappa(\lambda)$ by*

$$m(A) = \{\xi | A \in X_\xi\}\ , \qquad A \in S_\kappa(\lambda)\ .$$

Then m satisfies the conditions of Proposition 3.3.

Proof. Let F be a function $S_\kappa(\lambda) \to \lambda$ so that for a.e. A, $F(A) \in m(A)$. If F is not constant $(\mathrm{mod}\, D)$, then for every $\xi < \lambda$,

$$T_\xi = X_\xi \cap \{A | F(A) \neq \xi\} \in D\ .$$

By unique regularity, there is a $P \in D$ so that for every $\xi < \lambda$, $P \cap P_\xi \subseteq T_\xi$; i.e., for $A \in P$,

$$A \in X_\xi \to F(A) \neq \xi \qquad (\xi < \lambda)\ .$$

Thus, if $A \in P$,

$$A \in X_{F(A)} \to F(A) \neq F(A)\ ,$$

a contradiction, since for any A, $A \in X_{F(A)}$.

Propositions 3.3 and 3.4 immediately yield the following theorem.

Theorem 3.5. *Let D be an ultrafilter over $S_\kappa(\lambda)$, j the associated elementary embedding of V into V^D. Then the following are equivalent:*

(1) *D is uniquely (κ, λ)-regular.*

(2) *There is a map $m: S_\kappa(\lambda) \to S_\kappa(\lambda)$ so that for every $\xi < \lambda$, $\{A | \xi \in m(A)\} \in D$, and so that if G is a function $S_\kappa(\lambda) \to \lambda$ so that for a.e. A, $G(A) \in m(A)$, then G is constant* $(\mathrm{mod}\, D)$.

(3) *There is a $x \in V^D$ so that for $y \in V^D$, $y \in x$ in V^D if and only if $y = j(\xi)$ in V^D for some $\xi < \lambda$ and $x \subseteq j(S_\kappa(\lambda))$ in V^D.*

Corollary 3.6. *If D is a non-principal κ-complete ultrafilter over κ, then D is uniquely (κ, κ)-regular.*

Theorem 3.7. (1) *If D is a normal ultrafilter over $S_\kappa(\lambda)$ so that for every $\xi < \lambda$, $S_\kappa(\xi) \notin D$, then D is uniquely (κ, λ)-regular.*

(2) *If D is a uniquely (κ,λ)-regular ultrafilter, then there is a normal ultrafilter U over $S_\kappa(\lambda)$ so that $U \leq D$ and U is uniquely (κ,λ)-regularized by the family $\{P_\xi \mid \xi < \lambda\}$.*

Proof. (1) If D is normal over $S_\kappa(\lambda)$, let $A = \{\xi \mid P_\xi \in D\}$. Then $S_\kappa(A) \in D$, and if ϕ is a one-to-one onto function $\lambda \to A$, then ϕ induces an isomorphism between D and a normal ultrafilter U over $S_\kappa(\lambda)$ so that for every $\xi < \lambda$, $P_\xi \in U$.

(2) If $\{X_\xi \mid \xi < \lambda\}$ is the uniquely (κ,λ)-regularizing set for D, in the function of Theorem 3.5, define U by

$$X \in U \leftrightarrow m^{-1}(X) \in D\ .$$

Thus,

$$m^{-1}(P_\xi) = \{A \mid \xi \in m(A)\} = X_\xi\ ,$$

so that for every $\xi < \lambda$, $P_\xi \in U$. Also, U is normal; for if f is a function so that for a.e. A, $F(A) \in A$, define G by

$$G(A) = F(m(A))\ .$$

Then there is a $\xi < \lambda$ so that

$$\{A \mid G(A) = \xi\} = m^{-1}(\{A \mid F(A) = \xi\}) \in D\ .$$

We give another criterion for unique regularity:

Theorem 3.8. *Suppose that D is uniform and countably complete over λ. Then D is uniquely (κ,λ)-regular if and only if $X < j(\kappa)$ in V^D, where j is the associated elementary embedding into V^D, and X the function representing λ, and there are partial functions $f_\xi : \lambda \to V$ for $\xi < \lambda$ satisfying the following three conditions:*

(1) *Each f_ξ is defined a.e.*

(2) *If $\xi \neq \eta$, then* range $(f_\xi) \cap$ range $(f_\eta) = 0$.

(3) *There is a function representing $\{[f_\xi]_D \mid \xi < \lambda\}$ in V^D.*

Proof. For if we are given such f_ξ's, let F be the representing function of part (3) of our conditions, so that for every α, $|F(\alpha)| < \kappa$. Let

$$m(\xi) = \{\eta \mid f_\eta(\xi) \text{ defined and } f_\eta(\xi) \in F(\xi)\} .$$

Then the conditions of Theorem 3.5 are fulfilled.

As a direct consequence of Theorem 3.5, we get the following result, which should be compared with Theorem 0.6.

Theorem 3.9. *If for every regular $\lambda > \kappa$ there is a κ-complete, (κ, λ)-uniquely regular ultrafilter, κ is supercompact.*

In Section 5 of this paper, we shall construct a regular but not uniquely regular ultrafilter.

4. Normal ultrafilters

One can, of course, construct other embedding axioms than supercompactness. The following one was formulated by K. Kunen who proved that it implies the existence of non-trivial non-regular ultrafilters.

Definition 4.1. A regular cardinal κ is *huge* if there is an elementary embedding j of V into a transitive submodel M so that $j|R(\kappa) = \mathrm{id}$, $j(\kappa) \gtrsim \kappa$, and $M^{j(\kappa)} \subseteq M$.

Now assume that we are in the situation of Definition 4.1. Then, if $\lambda = j(\kappa)$ and $G = \{j(\xi) \mid \xi < \lambda\}$, the ultrafilter U defined by

$$X \in U \leftrightarrow G \in j(X)$$

is κ-complete and concentrated on $S_{\kappa^+}(\lambda)$. Moreover, U is normal over $S_{\kappa^+}(\lambda)$. By Theorem 1.13, U is not (κ, λ)-regular.

We can get a somewhat sharper result: Let λ be the least cardinal $> \kappa$ so that there is a κ-complete normal ultrafilter U over $S_{\kappa^+}(\lambda)$. For every $\alpha < \lambda$, the map

$$A \to A \cap \alpha \qquad (A \in S_{\kappa^+}(\lambda))$$

must then project U to a normal ultrafilter over $S_\kappa(\lambda)$. Hence, for every $\alpha < \lambda$,

$$Q_\alpha = \{A \mid |A \cap \alpha| < \kappa\} \in U .$$

Thus, the "diagonal intersection" of Q_α's, the set

$$Q = \{A \in S_{\kappa^+}(\lambda) \mid \alpha \in A \to A \in Q_\alpha\}$$

belongs to U. Hence, for a.e. A,

$$\alpha \in A \to |A \cap \alpha| < \kappa \ .$$

Thus, the first function greater than all constants $< \lambda$ has cofinality $j(\kappa)$. Hence

$$\operatorname{cof}(\Pi_U \kappa) = \lambda ,$$

and since U is obviously (κ^+, λ)-regular,

$$\mathrm{cof}\,(\Pi_U \kappa^+) \geq \lambda^+ .$$

The relationship between non-(κ, λ)-regular ultrafilters and supercompact cardinals is unclear. The following observations can, however, be made:

Assume κ is supercompact, $\lambda \geq \kappa$ regular and $\rho \geq 2^\lambda$ regular. Let $\{X_\gamma^\delta \mid \gamma < \lambda\}, \delta < \rho$, enumerate all (κ,λ)-regularizing families in $P(\lambda)$. For $A \in S_\kappa(\rho)$, define

$$T(A) = \bigcup_{\xi \in A} \bigcap_{\gamma \in A \cap \lambda} X_\gamma^\xi .$$

Let $\mathfrak{N}$ denote the set of all normal κ-complete ultrafilters U over $S_\kappa(\rho)$ so that for every $\xi < \rho$, $P_\xi \in U$. By Theorem 2.1, each $U \in \mathfrak{N}$ is λ-universal.

Theorem 4.2. *There is a uniform, κ-complete, non-(κ, λ)-regular ultrafilter over λ if and only if there is a $U \in \mathfrak{N}$ so that*

$$\Sigma \overset{\text{def}}{=} \{A \mid T(A) \supseteq \lambda \setminus \sup(A \cap \lambda)\} \in U$$

if and only if for every $U \in \mathfrak{N}$, $\Sigma \in U$.

Proof. Suppose that D is κ-complete uniform non-(κ, λ)-regular over λ. Let F be a function $S_\kappa(\rho) \to \lambda$ so that

$$X \in D \leftrightarrow F^{-1}(X) \in U$$

for some $U \in \mathfrak{N}$. Since D is uniform, for a.e. A, $\lambda > F(A) \geq \sup(A \cap \lambda)$. Also, for a.e. A, $F(A) \notin T(A)$; for if not, then for a.e. A there is a $G(A) \in A$ so that

$$F(A) \in \bigcap_{\gamma \in A \cap \lambda} X_\gamma^{G(A)}.$$

Now G is a constant ξ_0 (mod U). Thus for every $\gamma < \lambda$, $F(A) \in X_\gamma^{\xi_0}$ for a.e. A; i.e. $F^{-1}(X_\gamma^{\xi_0}) \in U$. Hence $\{X_\gamma^{\xi_0} \mid \gamma < \lambda\} \subseteq D$, a contradiction.

Conversely suppose that there is a $U \in \mathfrak{N}$ so that $\Sigma \in U$. Define a map $F\colon S_\kappa(\rho) \to \lambda$ by

$$F(A) = \mu\xi\ [\xi \geq \sup(A \cap \lambda),\ \xi \notin T(A)]$$

and a uniform ultrafilter D over λ by

$$X \in D \leftrightarrow F^{-1}(X) \in U\ .$$

Then D is not (κ, λ)-regular, since for a fixed $\xi < \rho$, we have for a.e. A,

$$F(A) \notin \bigcap_{\gamma \in A \cap \lambda} X_\gamma^\xi\ ,$$

i.e., there is a $t(A) \in A \cap \lambda$ so that for a.e. A,

$$F(A) \notin X_{t(A)}^\xi\ .$$

By normality, t must be a constant $\gamma_0 (\mathrm{mod}\ U)$. Thus, $F^{-1}(X_\gamma^\xi) \notin U$, i.e., $X_\gamma^\xi \notin D$.

As an example of a situation where Σ is "big" in every $U \in \mathfrak{N}$, we have hence all cardinals λ which are measurable.

Regularity can also be characterized in the following fashion:

Theorem 4.3. *Let $\rho \geq \lambda \geq \kappa$ be regular cardinals, U normal over $S_\kappa(\rho)$ containing all the P_ξ's, $\pi: S_\kappa(\rho) \to S_\kappa(\lambda)$ be defined by*

$$\varphi(x) = X \cap \lambda\ .$$

Let D be a κ-complete uniform ultrafilter over λ given from U by the projecting function $f: S_\kappa(\rho) \to \lambda$. Then D is (κ, λ)-regular if and only if there is a set $P \in U$ so that for any $\delta < \lambda$,

$$(*) \qquad |\varphi''(f^{-1}(\{\delta\}) \cap P)| < \kappa\ .$$

Proof. Assume that D is regularized by the family $\{X_\xi \mid \xi < \lambda\}$. We can find a set $P \in U$ so that for every $\xi < \lambda$

$$P \cap P_\xi \subseteq f^{-1}(X_\xi)$$

by unique regularity. Thus for any set $S \subseteq \lambda$ of cardinality κ,

$$\bigcap_{\xi \in S} f''(P \cap P_\xi) = 0\ ,$$

i.e., $(*)$ holds. Conversely, if $(*)$ holds, $\{f''(P \cap P_\xi) \mid \xi < \lambda\}$ is the desired regularizing family.

As a corollary to Theorem 4.3 we get the following theorem.

Let U be a κ-complete normal ultrafilter over $S_\kappa(\lambda)$ containing all the P_ξ's.

Theorem 4.4. *Given an ordinal* α *so that* $\operatorname{cof}(\alpha) \geq \kappa$, $\kappa \leq \alpha \leq \lambda$, *define an ultrafilter* D_α *over* α *by*

$$X \in D_\alpha \leftrightarrow \psi_\alpha^{-1}(X) \in U,$$

where $\psi_\alpha(A) = \sup(A \cap \alpha)$. *Then:*

(1) ψ_α *is the first function* (mod U) *greater than all constants* $< \alpha$.

(2) D_α *is a* κ-*complete,* (κ, $\operatorname{cof}(\alpha)$)-*regular ultrafilter over* α *and is concentrated on a set of cardinality* $\operatorname{cof}(\alpha)$.

(3) *If* $\operatorname{cof}(\alpha) = |\alpha|$, *there is a set* $P \in U$ *so that for every* $\alpha < \delta$ *there is a set* X_δ *of cardinality* $< \kappa$ *so that*

$$A \in P, \quad \psi_\alpha(A) = \delta \to A \cap \alpha \subseteq X_\delta .$$

(4) *If* $\psi = \psi_\lambda$, *there is a set* $Z \in U$ *so that for every* $\delta < \lambda$,

$$|Z \cap \psi^{-1}(\{\delta\})| < \kappa .$$

Robert Solovay has improved (4) and shown that this ψ is actually 1–1 on a set $Z \in U$. In particular, any normal ultrafilter over $S_\kappa(\lambda)$ is then $\leq$-minimal.

5. Products of ultrafilters

In the following, fix U, V to be κ-complete uniform ultrafilters over λ, μ, respectively, and let κ, λ, μ, ρ be regular cardinals so that $\omega < \kappa \leq \rho$. We say that an ultrafilter U is ρ-uniquely regular if it is (ρ, ρ)-uniquely regular. It is then trivial to check that U is (κ, ρ)-uniquely regular if and only if U is (κ, ρ)-regular and ρ-uniquely regular.

Proposition 5.1. *If $U \times V$ is ρ-uniquely regular, so is U.*

Proof. Let $\{X_\xi \mid \xi < \rho\}$ be a ρ-uniquely regularizing family for $U \times V$. Then for every $\xi < \rho$,

$$S_\xi \overset{\text{def}}{=} \{\alpha \mid \; X_\xi \mid \alpha \in V\} \in U \;,$$

and $\{S_\xi \mid \xi < \rho\}$ is a ρ-uniquely regularizing family for U. Given a family $\{T_\xi \mid \xi < \rho\} \subseteq U$, consider the sets $Y_\xi = T_\xi \times \mu$ for $\xi < \rho$. There exists a $P \in U \times V$ so that for any $\xi < \rho$, $P \cap X_\xi \subseteq Y_\xi$ and

$$S = \{\alpha \mid \; P \mid \alpha \in V\} \in U \,.$$

Then, if $\gamma \in S \cap S_\xi$, $P \mid \gamma \in V$ and $X_\xi \mid \gamma \in V$. Hence,

$$(P \mid \gamma) \cap (X_\xi \mid \gamma) = (P \cap X_\xi \mid \gamma) \in V \,.$$

Now, since $P \cap X_\xi \subseteq Y_\xi$ for $\xi < \rho$, $Y_\xi \mid \gamma \in V$; so $\gamma \in T_\xi$. Hence, for any $\xi < \rho$,

$$S \cap S_\xi \subseteq T_\xi \,.$$

Proposition 5.2. *If U is uniquely (κ, ρ)-regular, so is $U \times V$.*

Proof. If $\{T_\xi \mid \xi < \rho\}$ uniquely ρ-regularizes U, then

$$Q_\xi = T_\xi \times \mu \qquad (\xi < \rho)$$

uniquely ρ-regularizes $U \times V$. Given a family $\{X_\xi \mid \xi < \rho\}$ of elements of $U \times V$, let

$$S_\xi = \{\alpha \mid \; X_\xi \mid \alpha \in V\} \qquad (\xi < \rho) \,.$$

We can without loss of generality assume that $X_\xi \subseteq Q_\xi$ (otherwise intersect) so that every α belongs to less than κ S_ξ's. Thus,

$$R_\alpha = \bigcap \{X_\xi | \alpha | \quad \alpha \in S_\xi\} \in V .$$

Let S be a set in U so that for every $\xi < \rho$, $S \cap T_\xi \subseteq S_\xi$, and let

$$T = \{(\alpha, \beta) | \beta \in R_\alpha\} \cap (S \times \mu) .$$

Then for every $\xi < \rho$, $Q_\xi \cap T \subseteq X_\xi$.

In general, we cannot deduce from the unique (κ, ρ)-regularity of $U \times V$ the unique (κ, ρ)-regularity of U. For assume that U is a normal κ-complete ultrafilter over $S_{\kappa^+}(\rho)$ and V a normal κ-complete ultrafilter over $S_\kappa(\rho)$. Then the family $\{P_\xi \times P_\xi | \xi < \rho\}$ uniquely (κ, ρ)-regularizes $U \times V$.

However, for "small" cardinals ρ the above deduction can be made, and we obtain:

Proposition 5.3. *Let ρ be a regular cardinal between κ and the first measurable cardinal $> \kappa$. Then, if U is a κ-complete, ρ-uniquely regular ultrafilter, then U is (κ, ρ)-uniquely regular.*

Proof. Assume this were not the case. By the methods of Section 4 we get a κ-complete ultrafilter D over $P(\rho)$ so that for every $\xi < \rho$, $P(\xi) \notin D$ and D is normal and $S_\kappa(\rho) \notin D$. Let j be the associated elementary embedding of U into a transitive submodel M of the universe. Then $M^{j(\kappa)} \subseteq M$ and hence $j(\kappa)$ is a measurable cardinal between κ and ρ.

As a corollary, we get

Theorem 5.4. (1) *$U \times V$ is uniquely (κ, κ^+)-regular if and only if U is.*

(2) *If V is non-(κ, κ^+)-regular and U is uniquely (κ, κ^+)-regular, then*

$$U \times V \not\equiv V \times U.$$

(3) *If U is normal κ-complete over $S_\kappa(\kappa^+)$ and D is κ-complete over κ, then $D \times U$ is (κ, κ^+)-regular but not uniquely (κ, κ^+)-regular.*

The behavior of regular ultrafilters under products is far more complicated. Let U be a fixed weakly normal κ-complete uniform ultrafilter over a regular cardinal λ, $\omega < \kappa \leq \lambda$, and let E_α be a κ-complete ultrafilter over a set X_α $(\alpha < \lambda)$. We define the U-sum D of the E_α's as follows: Let Γ be the set

$$\Gamma = \{(\alpha, \beta) \mid \alpha < \lambda, \beta \in X_\alpha\} .$$

A subset X of Γ belongs to D if and only if

$$\{\alpha \mid \ X \mid \alpha \in E_\alpha\} \in U .$$

Let $\bar{g}_\alpha$ be the first function greater than all constants $< \alpha$ on X_α (mod E_α). Define a function f on Γ by

$$f(\alpha,\beta) = \bar{g}_\alpha(\beta) , \qquad (\alpha, \beta) \in \Gamma .$$

Theorem 5.5. *f is the first function greater than all the constants $< \lambda$ (mod D).*

Proof. By the uniformity of U, f is greater than all the constants $< \lambda$ (mod D). If g is a function $\Gamma \to V$ so that $g < f$ (mod D), then

$$S = \{\alpha \mid \{\beta \mid g(\alpha,\beta) < \bar{g}_\alpha(\beta)\} \in E_\alpha\} \in D ,$$

i.e., for $\alpha \in S$ there is a $t_\alpha < \alpha$ so that

$$\{\beta \mid g(\alpha, \beta) < t_\alpha\} \in E_\alpha .$$

By weak normality of U, there is a $\xi < \lambda$ and a subset S' of S so that $S' \in U$ and $t_\alpha \leq \xi$ for $\alpha \in S'$. Hence $g \leq \xi$ (mod D).

Applying Theorem 1.6, we get:

Theorem 5.6. *Let D be the ultrafilter defined above. Then D is (κ, λ)-regular if and only if*

$$\{\alpha \mid E_\alpha \textit{ is } (\kappa, \operatorname{cof}(\alpha))\textit{-regular}\} \in U .$$

Proof. D is (κ, λ)-regular if and only if $\{(\alpha, \beta) \mid \operatorname{cof}(\bar{g}_\alpha(\beta)) < \kappa\} \in D$.

In particular, if E is a fixed κ-complete ultrafilter over a regular cardinal μ and we let each E_α be E, we get:

Theorem 5.7. *$U \times E$ is (κ, λ)-regular if and only if E is (κ, ν)-regular for a $\nu < \lambda$ so that $\{\alpha \mid \mathrm{cof}(\alpha) \leq \nu\} \in U$.*

As a corollary, we get:

Theorem 5.8. *If U is a κ-complete uniform ultrafilter over κ^+ and D a κ-complete ultrafilter over κ, then $U \times D$ is (κ, κ^+)-regular.*

The following result solves an open problem of K. Kunen:

Theorem 5.9. *Assume that $\kappa > \omega$ is a regular cardinal so that for every regular $\lambda \geq \kappa$ there is a κ-complete uniform ultrafilter over λ. Then κ is strongly compact.*

Proof. Let U_λ be a weakly normal uniform κ-complete ultrafilter over λ. By Theorem 0.6, it suffices to prove the following statement by induction on λ: If $\lambda > \kappa$ is a regular cardinal, then there is a κ-complete (κ, λ)-regular ultrafilter D_λ over λ.

The basis and the induction step $\lambda = \mu^+$, where μ is regular, is trivial. By Theorem 5.10, $U_\lambda \times D_\mu$ does the job.

In general, if the above statement is true for every $\alpha < \lambda$, λ regular, pick E_α to be κ-complete $(\kappa, \mathrm{cof}(\alpha))$-regular over a set X_α, and then let D_λ be the U_λ-sum of the E_α's. Then D_λ is (κ, λ)-regular.

Actually, we have the following theorem:

Theorem 5.10. *If $\kappa > \omega$ is strongly compact, then for every regular $\lambda > \kappa$ there is a uniform κ-complete, (κ, λ)-regular ultrafilter over λ.*

Using the above result and the fact that any κ-complete, (κ, λ)-regular ultrafilter is $(\kappa, \lambda^{\kappa})$-regular, Robert Solovay has proved the following remarkable result:

Theorem 5.11. (Robert Solovay). *If $\lambda > \kappa$ is regular and κ is strongly compact, then $\lambda^{\kappa} = \lambda$. The Generalized Continuum Hypothesis holds at any singular strong unit cardinal $> \kappa$ of cofinality κ.*

We wish to point out another, perhaps easier, way of proving Theorem 5.10. Assume that $\lambda \geq \kappa \geq \omega$ are regular so that every regular μ between λ, κ carries κ-complete uniform ultrafilters. Let $\mathcal{W}$ denote the set of all weakly normal κ-normal, κ-complete ultrafilters over λ. Define a partial ordering $\lhd$ on $\mathcal{W}$ as follows: Given $D, E \in \mathcal{W}$, then $D \lhd E$ if and only if for a.e. (mod E) α there is a countably complete ultrafilter E_α over α so that

$$X \in D \leftrightarrow \{\alpha \mid X \cap \alpha \in E_\alpha\} \in E .$$

Proposition 5.12. *If D is $\lhd$-minimal, then D is (κ, λ)-regular.*

Proof. For if D is not (κ, λ)-regular, then

$$X_0 = \{\alpha \mid \operatorname{cof}(\alpha) \geq \kappa\} \in D .$$

For any $\alpha \in X_0$, pick a weakly normal ultrafilter E_α over α. Define E by

$$X \in E \leftrightarrow \{\alpha \mid X \cap \alpha \in E_\alpha\} \in D .$$

Then $E \lhd D$; i.e., D is not $\lhd$-minimal.

Thus, to prove the existence of regular ultrafilters, it suffices to show that $\lhd$ is well-founded. In the case of λ being inaccessible this is quite easy: If $D \lhd E$, then $i_D(\lambda) < i_E(\lambda)$; if E_α is countably complete and

$$X \in D \leftrightarrow \{\alpha \mid X \cap \alpha \in E_\alpha\} \in E ,$$

then for any $f: \lambda \to \lambda$,

$$\{\alpha \mid f: \alpha \to \alpha\} \in E.$$

Hence

$$i_D(\lambda) = \text{o.t.}\,(\Pi_D \langle \lambda, < \rangle) \leq \text{o.t.}\,(\Pi_E \langle \text{o.t.}\, \Pi_{E_\alpha} \langle \alpha, < \rangle, < \rangle)$$

$$< \text{o.t.}\,(\Pi_E \langle 2^{\alpha^+}, < \rangle) < \text{o.t.}\,(\Pi_E \langle \lambda, < \rangle) = i_E(\lambda) .$$

In the case of λ not being inaccessible, we can apply the above argument to $1_\lambda(\kappa)$, as Kenneth Kunen pointed out.

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